Risk Management Project

VaR with heteroskedasticity and non-normality
(McNeil and Frey, 2000)

Jérôme van Eck Duymaer van Twist, Elyes Mahjoubi

May 2020

Abstract

The risk control, and more precisely the reduction of specific risks, is vital for different institutions, such as financial corporations or firms that require to reduce potential losses. The quantitative risk management gives statistical implementations to achieve the purposes cited before. In fact, the aim of this paper is to find a better estimation of quantiles of the returns distribution and the tail of innovation distribution related to the S&P500 and SMI indexes. The approaches and methodology are based on the Alexander J. McNeil and Frey researches. The first chapter introduces the dataset used and an explanation of mathematical instruments employed to approach the estimations. The second chapter gives a better view on the empirical results and methods applied on the financial return series. Finally, the conclusion chapter exhibits the final comparison between using the VaR (Value at Risk) or Expected shortfall and which of the distribution fits better the value-at-risk estimation.

1 Data and Methodology

The dataset consists of closed prices of S&P500 and SMI indexes reached from the Reuters and Yahoo. The two samples take into account the dividends payments and have a daily frequency that goes from January 1991 to mid-May 2020. Each daily VaR were computed on a 10 days basis like the bale commitment state.

In order to estimate VaR and expected shortfall, which describe the tails of conditional distribution of heteroskedastic sample of S&P500 and SMI indexes sample, the methodology chosen in this paper is a combination of quasi maximum likelihood fitting GARCH models and the extreme value theory. The use of the Quasi-Log Likelihood estimation (Maximum Likelihood estimation plus a robust Covariance estimator) and GARCH model is prompt to estimate conditional volatility, which will reveal the non conformity for the financial datasets. Meanwhile, the non parametric estimations, or in other words the historical simulations, are used to estimate the central part of the distribution. Finally, the estimation of the distribution of residuals generated by the GARCH model is made by the extreme value theory using a Generalized Pareto Distribution that we are going to discover across this paper.

We have:

$$X_t = \mu_t + \sigma_t Z_t$$

where X_t is a strictly stationary time series, which represents daily observation of negative log returns. We consider the innovations (Z_t) as a strict white noise (iid) with a mean of zero and a variance of one, as well as a marginal distribution $F_Z(z)$. μ_t and σ_t are assumed to be measurable with respect to the information about the return process available up to $t-1$ (G_{t-1}).

We are considering a measure of risk for the tail of a distribution, which is called the expected shortfall. The unconditional expected shortfall is defined as:

$$S_q = E[X|X > x_q]$$

and the conditional expected shortfall is:

$$S_q^t(h) = E[\sum_{j=1}^{h} X_{t+j}|\sum_{j=1}^{h} X_{t+j} > x_q^t(h), G_t]$$

with h being the number of days. As we are particularly interested in the quantiles and the expected shortfall for the 1-step predictive distribution (x_q^t and S_q^t). As we have :

$$F_{X_{t+1|G_t}}(x) = P(\sigma_{t+1}Z_{t+1} + \mu_{t+1} \leq x|G_t)$$

$$= F_Z(\frac{x - \mu_{t+1}}{\sigma_{t+1}})$$

which we can simplify in order to get:

$$x_q^t = \mu_{t+1} + \sigma_{t+1}z_q$$

$$S_q^t = \mu_{t+1} + \sigma_{t+1}E[Z|Z > z_q]$$

Where z_q is the upper qth quantile of the marginal distribution of Z_t which does not depend on t by assumption. In order to implement an estimation procedure for these values, we are going to use the GARCH(1, 1) process for the volatility. As for the dynamics of the conditional mean, we are going to use an AR(1) model. When we estimate x_q^t, we assume that the distribution of the innovation is standard normal which leads to the quantile of the innovation distribution to be : $z_q = \phi^{-1}(q)$, where $\phi(z)$ is the standard normal distribution function.

There is also another approach which is to assume that the innovations have a leptokurtic distribution (a Student t distribution for instance which is scaled in order to have a variance of 1). We suppose the following: $Z = \sqrt{\frac{v-2}{v}}T$ where T has a t-distribution on v >2 degrees of freedom with the density function $F_T(t)$. We then have $z_q = \sqrt{\frac{v-2}{v}}F_T^{-1}(q)$. t-innovations using a GARCH like model can be fitted using a maximum likelihood. However it has an additional parameter: the degree of freedom v which can be estimated. This approach yields good results if both the positive and negative tails are somewhat equal.

In order to estimate σ_{t+1} and μ_{t+1}, we use a constant memory (n), which will let us have a data set consisting of the last n negative log returns at the end of day t . These values are the realisations from an AR(1) - GARCH(1, 1) process. Therefore, the conditional variance of the mean-adjusted series $\epsilon_t = X_t - \mu_t$ which is given by:

$$\sigma_t^2 = \alpha - 1\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

where α_0>0, α_1>0 and β>0.
We then obtain the following conditional mean:

$$\mu_t = \phi X_{t-1}$$

The ϵ_t (mean-adjusted series) is strictly stationary when:

$$E[log(\beta + \alpha_1 Z_{t-1}^2)] < 0$$

We can ensure that the marginal distribution $F_x(x)$ has a finite second moment by using Jensen's inequality and the convexity of -log(x), which leads to a sufficient condition for the above equation: $\beta + \alpha_1 < 1$
We calculate the residuals in order to check the adequacy of the GARCH modelling as well as as for the EVT model. They are calculated in the following manner:

$$(z_{t-n+1}, ..., z_t) = (\frac{x_{t-n+1} - \hat{\mu}_{t-n+1}}{\hat{\sigma}_{t-n+1}}, ..., \frac{x_t - \hat{\mu}_t}{\hat{\sigma}_t})$$

The residuals are supposed to be iid if the fitted model is sustainable. If we consider the fitted model as satisfying, we can end the first part by calculating estimates of the conditional mean and variance for the day $t+1$:

$$\hat{\mu}_{t+1} = \hat{\phi} x_t$$

$$\hat{\sigma}_{t+1}^2 = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{\epsilon}_t^2 + \hat{\beta} \hat{\sigma}_t^2$$

where $\hat{\epsilon}_t = x_t - \hat{\mu}_t$

For the second part of the project, we have to fix a threshold u. We then assume that any excess residuals that are over u have a generalized Pareto distribution (GPD):

$$G_{\xi,\beta}(y) = \begin{cases} 1 - (1 + \frac{\xi y}{\beta})^{\frac{-1}{\xi}} & \text{if} \quad \xi \neq 0 \\ 1 - exp(\frac{-y}{\beta}) & \text{if} \quad \xi = 0 \end{cases}$$

with $\beta > 0$, the support $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq \frac{-\beta}{\xi}$ when $\xi < 0$

Let $F_u(y)$ be an excess distribution above the chosen threshold u :

$$F_u(y) = P\{X - u \leq y | X > u\} = \frac{F(y+u) - F(u)}{1 - F(u)}$$

with $0 \leq y < x_0$, where x_0 is the right endpoint of F.

Now we can consider an equality point for $x > u$ in the tail of F.

$$1 - F(x) = (1 - F(u))(1 - F_u(x - u))$$

We can now obtain the tail estimator by estimating the first term on the right of the above equation using a random proportion of the data in the tail $\frac{N}{n}$, as well as estimating the second term by approximating the excess distribution with a GPD that we fit with a maximum likelihood for $x > u$

$$\hat{F}(x) = 1 - \frac{N}{n}(1 + \hat{\xi}\frac{x-u}{\hat{\beta}})^{\frac{-1}{\hat{\xi}}}$$

We then fix the number of data in the tail to $N = k$ with $k << n$. This allows us to obtain a random threshold at the (k+1)th order statistic. We then order the residuals $(z_{(1)} \geq z_{(2)} \geq ... \geq z_{(n)})$. The GPD parameters ξ and β are fitted to the data and we obtain the excess amounts over the threshold for all residuals which were exceeding the threshold : $(z_{(1)} - z_{(k+1)}, ..., z_{(k)} - z_{(k+1)})$ which gives us the tail estimator for $F_z(z)$ of the form:

$$\hat{F}_z(z) = 1 - \frac{k}{n}(1 + \hat{\xi}\frac{z - z_{k+1}}{\hat{\beta}})^{\frac{-1}{\hat{\xi}}}$$

when $q > 1 - \frac{k}{n}$ we are allowed to invert the formula, which gives us:

$$\hat{z}_q = \hat{z}_{q,k} = z_{k+1} + \frac{\hat{\beta}}{\hat{\xi}}((\frac{1-q}{\frac{k}{n}})^{-\hat{\xi}} - 1)$$

When the data is more symmetric, the t-distribution works better. It can even be considered as a special case of the method we are doing. This distribution is a heavy-tailed distribution whose limiting excess distribution is a GPD with ξ>0 which was characterized by Gnedenko (1943) as the following:

$$1 - F(x) = x^{\frac{-1}{\xi}} L(x)$$

With:

- $L(x)$ being a slowly varying function
- ξ the positive limiting parameter of the GPD

Also $\frac{1}{\xi}$ is the tail index of F. When we have a t-distribution with v degrees of freedom, the tail satisfies the following:

$$1 - F(x) \sim \frac{v^{\frac{v-2}{2}}}{B(\frac{1}{2}, \frac{v}{2})} x^{-v}$$

B(.) is the beta function. This gives us a symmetric distribution that reciprocates the value ξ with the degrees of freedom.

Next, we are going to use the Hill estimator in order to compare the GPD approach. The Hill estimator (Hill, 1975), which has been designed for heavy-tailed distribution data when we acknowledge the representation with ξ>0. The estimator for ξ which is based on the k in excess of the (k+1)th order statistic is:

$$\hat{\xi}^{(H)} = \hat{\xi}_k^{(H)} = k^{-1} \sum_{j=1}^{k} log z_{(j)} - log z_{k+1}$$

Given this, the associated quantile estimator is the following:

$$\hat{z_q}^{(H)} = \hat{z_{q,k}}^{(H)} = z_{(k+1)}(\frac{1-q}{\frac{k}{n}})^{-\hat{\xi}^{(H)}}$$

The next part consists in estimating the expected shortfall in our model. We remember from earlier the conditional expected shortfall:

$$S_q^t = \mu_{t+1} + \sigma_{t+1} E[Z|Z > z_q]$$

In order to estimate this, we need to estimate the expected shortfall for the innovation distribution. We use W, a random variable with a GPD distribution with the following parameters:

- $\xi < 0$
- β

We can then verify that :

$$E[W|W > w] = \frac{w + \beta}{1 - \xi}$$

with $\beta + w\xi > 0$. Now we assume that the excess values over the threshold u will have this exact distribution, for example $Z - u|Z > u \sim G_{\xi,\beta}$. When $z_q > u$, we get :

$$Z - z_q|Z > z_q = (Z - u) - (z_q - u)|(Z - u) > (z_q - u)$$

we can then show that the excess over the threshold z_q have a GPD distribution:

$$Z - z_q|Z > z_q \sim G_{\xi,\beta+\xi(z_q-u)}$$

with the same shape parameter ξ , however it has a different scaling parameter. By using the above equations, we can obtain:

$$E[Z|Z > z_q] = z_q(\frac{1}{1-\xi} + \frac{\beta - \xi u}{(1-\xi)z_q})$$

These equations finally allows us to get the conditional expected shortfall estimate by replacing the values in the formula seen above with the unknown values by the estimates based on GPD and replacing u with $z_{(k+1)}$. We then obtain:

$$\hat{S_q^t} = \hat{u}_{t+1} + \hat{\sigma}_{t+1}\hat{z_q}(\frac{1}{1-\hat{\xi}} + \frac{\hat{\beta} - \hat{\xi} z_{k+1}}{(1-\hat{\xi})\hat{z_q}})$$

We can finally get the shortfall to quantile ratio:

$$\frac{S_q^t}{x_q^t} \approx \frac{S_q^t - \mu_{t+1}}{x_q^t - \mu_{t+1}} = \frac{E[Z|Z > z_q]}{z_q}$$

This gives us the relation between the conditional one step quantiles and the shortfalls of the return process when μ_{t+1} is small.

For the last part, we have used a few tests in order to compare the methods we have used.

First, the binomial test, which compares the observed number of exceptions with the expected number of exceptions. We can do so using the properties of the binomial distribution with a test stat z such as:

$$z = \frac{x - Np}{\sqrt{Np(1-p)}}$$

where:

- x is the number of failures
- N is the number of observations
- p = 1-VaR level

This binomial test is approximately distributed as a standard normal distribution.

The second test we are using is the Kupiec's POF test. It is a variation on the binomial test and it stands for proportion of failures. It works with the binomial distribution approach and it uses a likelihood ratio in order to test if the probability of exceptions and the probability implied by the VaR confidence level are synchronized. The VaR model is rejected if the data shows that the probability of exceptions is different than p. We use the following POF statistic test:

$$LR_{POF} = -2log(\frac{(1-p)^{N-x}p^x}{(1-\frac{x}{N})^{N-x}(\frac{x}{N})^x})$$

Then we use the Christoffersen's test, which measures if the probability of observing an exception on a given day depends on the occurrence of an exception. This test measures the dependency between consecutive days only. The test stat for this test is :

$$LR_{CCI} = -2log(\frac{(1-\pi)^{n00+n10}\pi^{n01+n11}}{(1-\pi_0)^{n00}\pi_0^{n01}(1-\pi_1)^{n10}\pi_1^{n11}})$$

with:

- n00 being the number of periods with no failures followed by a period with no failures
- n10 is the number of periods with failures followed by a period with no failures
- n01 is the number of periods with no failures followed by a period with failures

- n11 being the number of periods with failures followed by a period with failures
- π_0 is the probability of having a failure on period t, given that no failure occurred on period t-1 = $\frac{n01}{n00+n01}$
- π_1 is the probability of having a failure on period t, given that a failure occurred on period t-1 = $\frac{n11}{n10+n11}$
- π is the probability of having a failure on period t = $\frac{n01+n11}{n00+n01+n10+n11}$

This test statistic is distributed asymptotically as a chi-square with 1 degree of freedom. It is possible to combine this statistic with the POF test in order to get a conditional coverage mixed test in the following manner:

$$LR_{cc} = LR_{POF} + LR_{CCI}$$

This test is also asymptotically distributed as a chi-square, however it has 2 degrees of freedom.

In order to calculate the VaR for the normal distribution, we suppose $L \sim N(\mu, \sigma^2)$. Then the VaR is computed as the following:

$$VaR_\alpha = \mu + \sigma\Phi^{-1}(\alpha)$$

where $\phi(.)$ is the standard normal CDF. This allows us then to calculate the expected shortfall with a normal distribution:

$$ES_\alpha = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$$

with $\phi(.)$ being the PDF of the standard normal distribution.

As for the VaR using a t-distribution, we assume $L \sim t(v, \mu, \sigma^2)$. $\frac{L-\mu}{\sigma}$ would have a standard t distribution with v>2 degrees of freedom. The VaR is calculated in the following manner:

$$VaR_\alpha = \mu + \sigma t_v^{-1}(\alpha)$$

where t_v is the CDF for the t distribution with v degrees of freedom. If we let $L \sim t(v, \mu, \sigma^2)$ such that $\tilde{L} := \frac{L-\mu}{\sigma}$ has a standard t distribution with v>2 degrees of freedom. We can then see that $ES_\alpha(L) = \mu + \sigma ES_\alpha(\tilde{L})$. This allows us to obtain the following:

$$ES_\alpha(\tilde{L}) = \frac{g_v(t_v^{-1}(\alpha))}{1-\alpha}(\frac{v + (t_v^{-1}(\alpha))^2}{v-1})$$

with $t_v(.)$ and $g_v(.)$ being the CDF and PDF of a standard t distribution with v degrees of freedom

2 Results

After transforming prices in daily log returns and assuming their dynamics as a linear combination μ and σZ_t, the estimation of the conditional quantiles is done, assuming a predictive distribution of returns over 252 days.

The first phase is the estimation of AR(1) and GARCH(1,1) models for the conditional mean and conditional volatility respectively. For the GARCH model, it is also useful to assume that the Z_t are normally standardize distributed in order to fit the model using the quasi-maximum-likelihood approach and obtain consistent parameters. So as to predict μ_{t+1} and σ_{t+1}, it is needed to implement the AR(1)-GARCH(1,1), for each historical, t-student distribution and normal conditional distribution.

		Coeff	t-value	p-value
SMI	Const	0.0589	5.929	3.050e-09
	Coeff	0.0329	2.491	1.272e-02
S&P500	Const	0.0623	6.831	8.447e-12
	Coeff	-0.0263	-2.136	3.271e-02

Table 1: AR(1) mean estimation for Normal distribution

		Coeff	t-value	p-value
SMI	Ω	0.0447	3.263	1.104e-03
	α	0.1359	8.612	7.195e-18
	β	0.8271	35.798	1.190e-280
S&P500	Ω	0.0181	4.790	1.667e-06
	α	0.1076	8.989	2.503e-19
	β	0.8776	70.061	0.000

Table 2: GARCH(1,1) volatility estimation for Normal distribution

		Coeff	t-value	p-value
SMI	Const	0.0697	7.406	1.300e-13
	Coeff	0.0193	1.635	0.102
S&P500	Const	0.0734	8.985	2.588e-19
	Coeff	-0.0369	-3.324	8.883e-04

Table 3: AR(1) mean estimation for t-student distribution

		Coeff	t-value	p-value
SMI	Ω	0.0268	6.038	1.562e-09
	α	0.1206	10.822	2.710e-27
	β	0.8587	69.385	0.000
	ν	7.8298	9.940	0.000
S&P500	Ω	0.0101	4.320	1.562e-05
	α	0.0991	9.843	7.362e-23
	β	0.8975	89.482	0.000
	ν	5.9251	13.911	0.000

Table 4: GARCH(1,1) volatility estimation for t-student distribution

The following graphs show the main market crashes for the SMI and S&P500 indexes. Comparing the two conditional volatility, the SMI tends to have more high peaks rather than the S&P500. For both indexes, residuals are consistent with the conditional volatility estimated from the model.

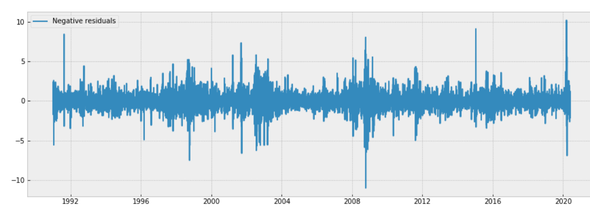

Figure 1: SMI negative residuals derived from AR(1)-GARCH(1,1) model under normal distribution

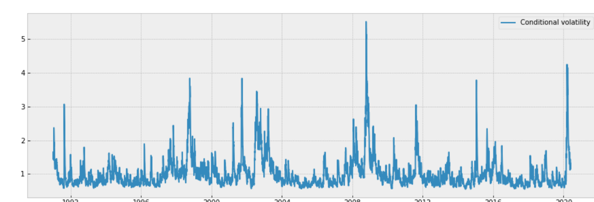

Figure 2: SMI conditional volatility from AR(1)-GARCH(1,1) model under normal distribution

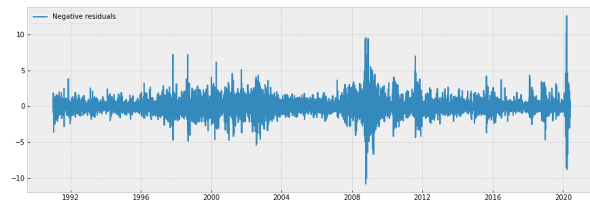

Figure 3: S&P500 negative residuals derived from AR(1)-GARCH(1,1) model under normal distribution

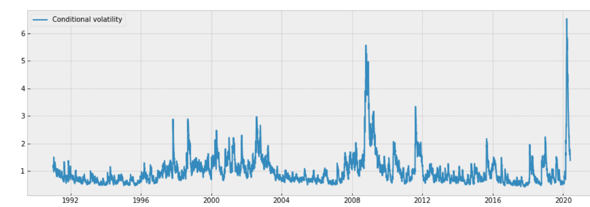

Figure 4: S&P500 conditional volatility from AR(1)-GARCH(1,1) model under normal distribution

Furthermore, the auto correlations for the absolute and relative values of the time series for the returns of each index and their residuals are showed in graphs from 5 to 12. We can see that there is auto correlations for the absolute returns but not for the absolute residuals, the i.i.d. condition seems to state only for the residuals.

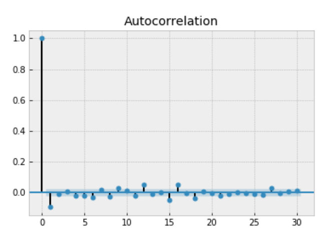

Figure 5: Autocorrelation of S&P500 returns

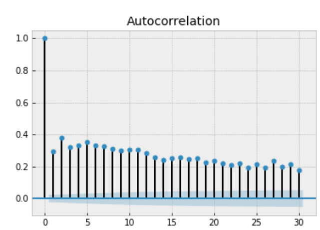

Figure 6: Autocorrelation of S&P500 absolute returns

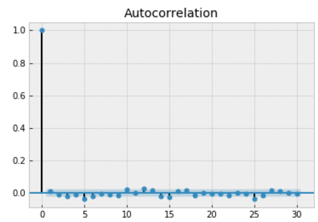

Figure 7: Autocorrelation of S&P500 residuals

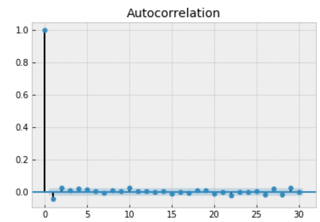

Figure 8: Autocorrelation of S&P500 absolute residuals

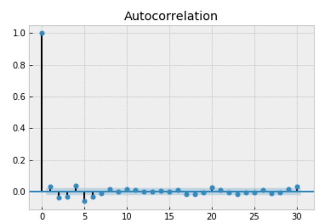

Figure 9: Autocorrelation of SMI returns

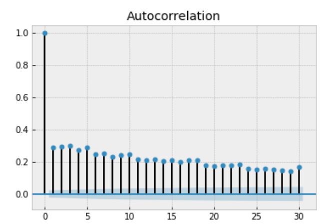

Figure 10: Autocorrelation of SMI absolute returns

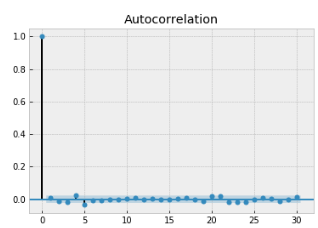

Figure 11: Autocorrelation of SMI residuals

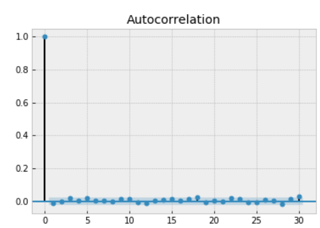

Figure 12: Autocorrelation of SMI absolute residuals

We confirmed the i.i.d hypothesis by the results from the Ljung-Box test. The test shows up that in all four first lags: only for the residuals,the hypothesis of i.i.d is not rejected in contrary to with the returns.

Lags	LB statistic	p-value
1	2.73	0.4365
2	3.36	0.567
3	4.54	0.282
4	4.87	0.294

Table 5: Ljung-Box test for S&P500

Lags	LB statistic	p-value
1	1.0519	0.305
2	2.506	0.289
3	5.567	0.136
4	10.550	0.033

Table 6: Ljung-Box test for SMI

The quantile-quantile plots below show for each index that residuals do not follow the normal distribution, especially for the S&P500 which manifests higher gap from the normal quantile distribution for the negative values of the tail and the opposite for the positive. Overall, they present a fatter tail's distribution. The normality is then not an appropriate assumption.

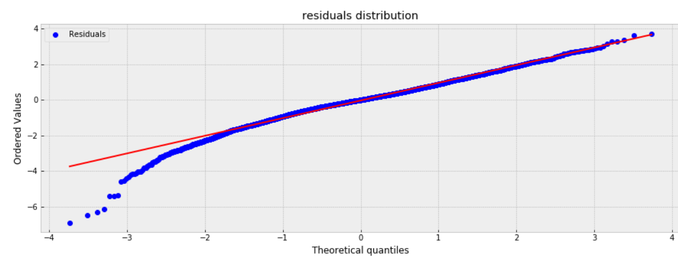

Figure 13: Q-Q plot of residuals versus normal quantile distribution S&P500

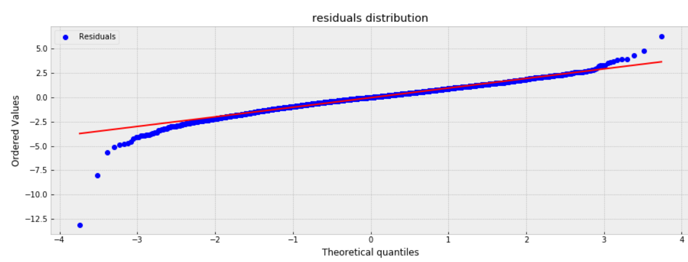

Figure 14: Q-Q plot of residuals versus normal quantile distribution SMI

The Hill estimator is the most useful parameter related to the tail behaviour. It allows to make an efficient decision about the k of the threshold, in this paper we used $k = 10\%$ of the sample. We then can estimate the ξ parameter, which represents the shape of the tail of a Pareto distribution as a power of its function, but it is not the best one to estimate the quantile below $q = 99\%$, which is better done by the GPD (Generalized Pareto Distribution).
The graph below shows the Hill estimation for the S&P500 fitting the empirical

data and the innovations. When we increase the threshold number, the hill's estimator becomes less performing. For this reason, the optimal k is around less than 300 for the S&P500 innovations.

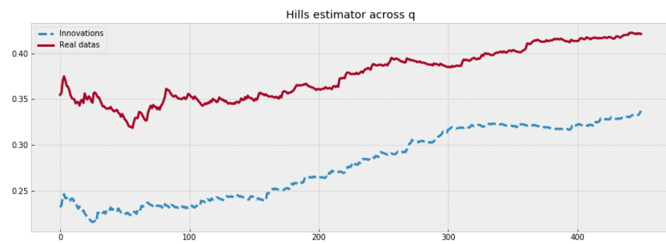

Figure 15: S&P500 Hill estimator across q

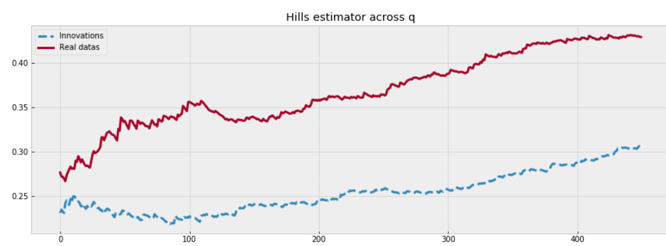

Figure 16: SMI Hill estimator across q

We can conclude that, Hill's estimator used for estimating the EVT's quantiles could be an efficient estimator at least as the GPD for $q >= 99\%$.

Following the extreme value theory and because the residuals has a fat tails, the Pareto's seems to be the appropriate distribution rather than the normal one, which usually underestimates the extreme values. Most of all, the generalized Pareto distribution is assumed to be modelling the tails of residuals over a threshold of 10% of the values by sorting from the lower to the higher. e.g. modelling excess distribution function. Looking at the figures 18 and 17, it is clear that the distributions of the residuals tails are approximately the same as the theoretical Generalized Pareto Cumulative distribution.

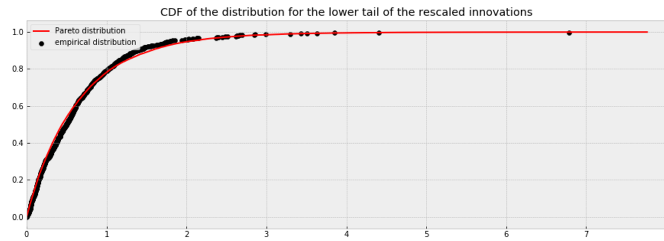

Figure 17: CDF of the SMI lower tail innovations vs the theoretical cdf of a GDP

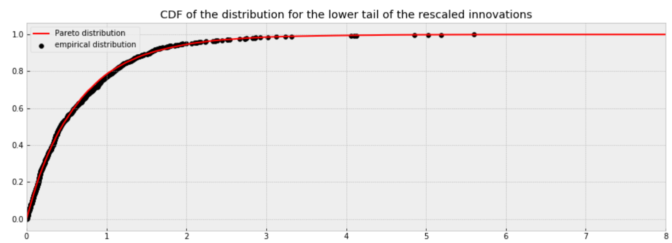

Figure 18: CDF of S&P500 lower tail innovations vs the theoretical cdf of a GDP

Finally, computing the extreme shortfall, known also as the Conditional value at risk which is obtained by taking the weighted average of extreme tails of the distribution of the expected residuals. ES is an alternative risk measure to VaR that grants more information about the size of extreme values of the distribution, particularly the quantile of the time series losses.

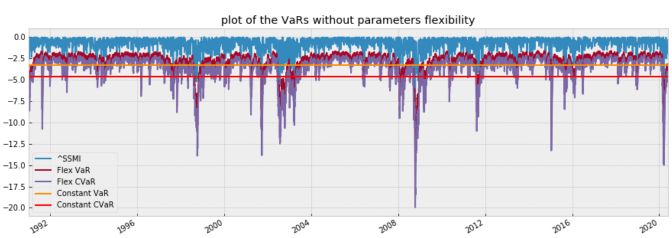

Figure 19: Var and ES of SMI with historical distribution(In sample and at 99%)

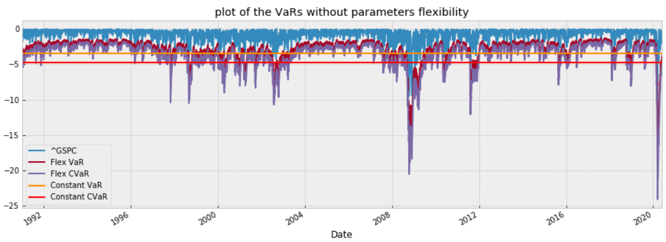

Figure 20: VaR and ES of S&P500 with GDP distribution(In sample and at 99%)

Figures above show the extreme negative values of the historical distribution obtained by AR(1)-GARCH(1,1) taking constant parameters. Meanwhile, the following graphs are made in Sample. Below we will see what are the results if we use the EVT conditional, the student-t conditional, the normal conditional and EVT unconditional.

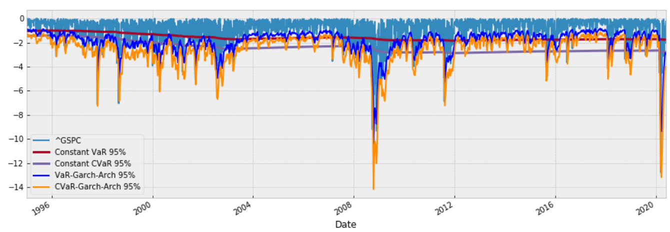

Figure 21: VaR and ES of S&P500 with Generalized Pareto distribution at 95%

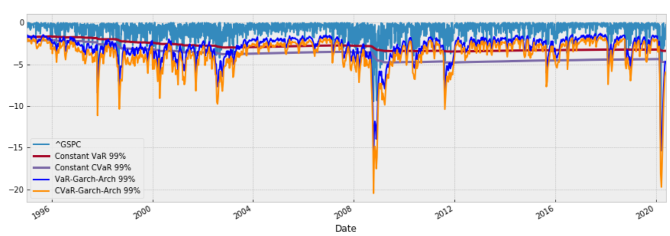

Figure 22: VaR and ES of S&P500 with Generalized Pareto distribution at 99%

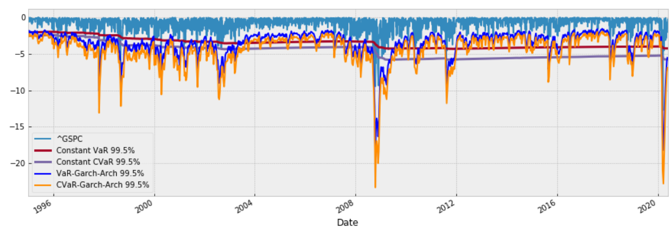

Figure 23: VaR and ES of S&P500 with Generalized Pareto distribution at 99.5%

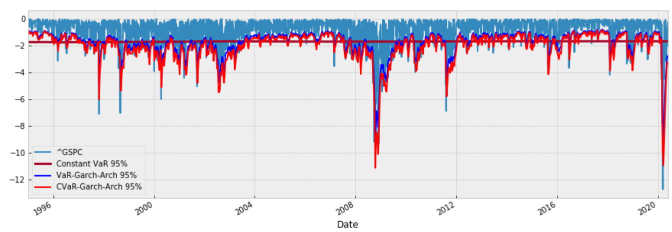

Figure 24: VaR and ES of S&P500 with conditional normal distribution at 95%

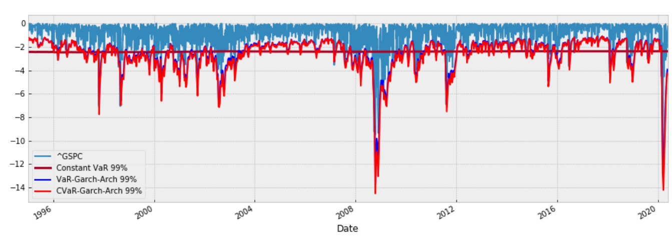

Figure 25: VaR and ES of S&P500 with conditional normal distribution at 99%

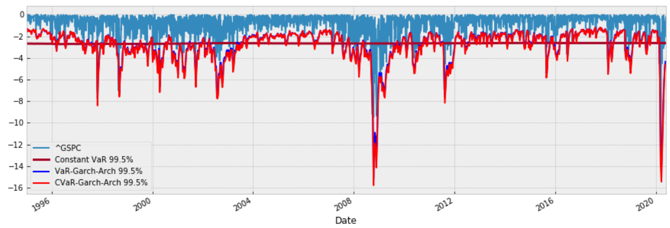

Figure 26: VaR and ES of S&P500 with conditional normal distribution at 99.5%

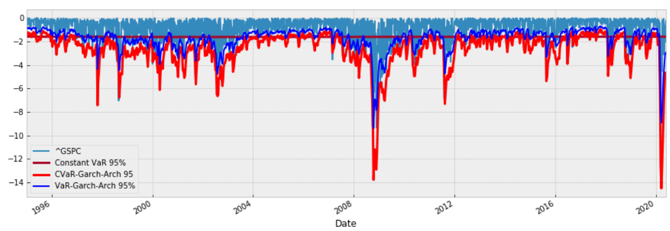

Figure 27: VaR and ES of S&P500 with t-student distribution at 95%

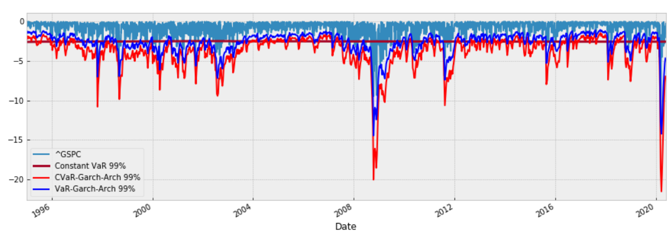

Figure 28: VaR and ES of S&P500 with t-student distribution at 99%

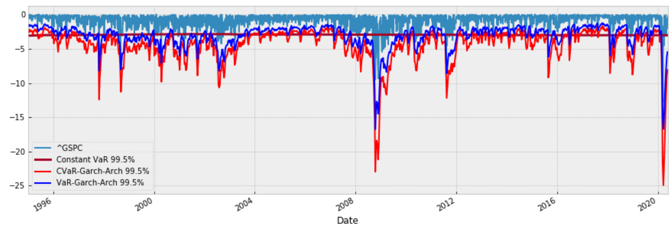

Figure 29: VaR and ES of S&P500 with t-student distribution at 99.5%

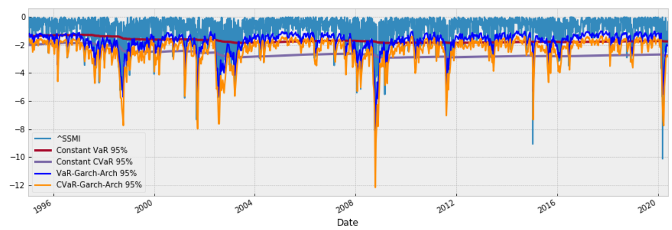

Figure 30: VaR and ES of SMI with Generalized Pareto distribution at 95%

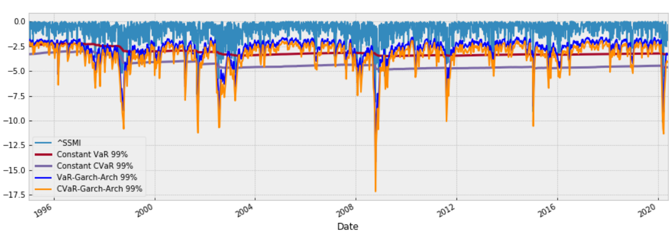

Figure 31: VaR and ES of SMI with Generalized Pareto distribution at 95%

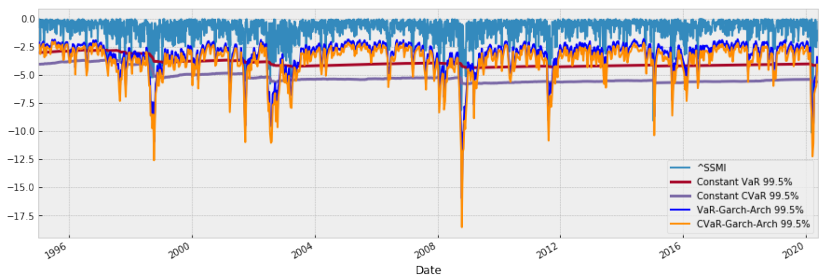

Figure 32: VaR and ES of SMI with Generalized Pareto distribution at 99.5%

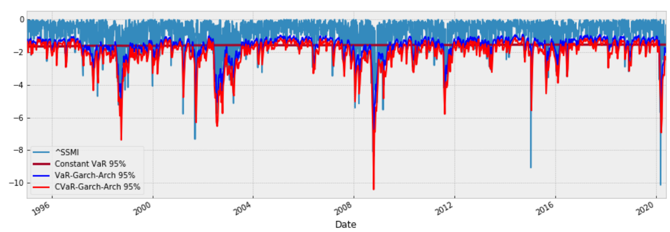

Figure 33: VaR and ES of SMI with conditional normal distribution at 95%

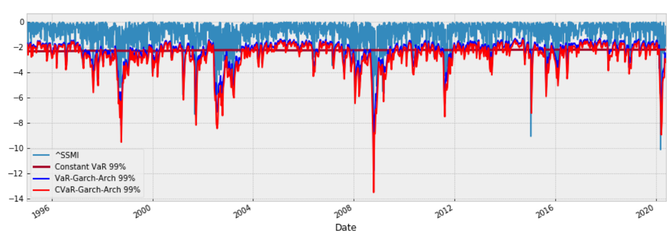

Figure 34: VaR and ES of SMI with conditional normal distribution at 99%

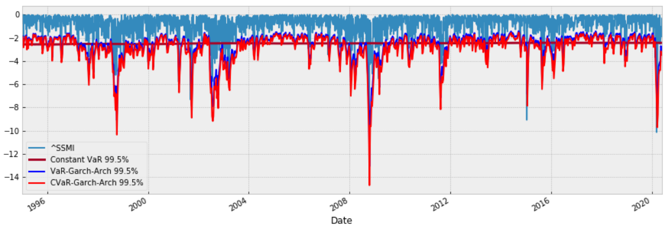

Figure 35: VaR and ES of SMI with conditional normal distribution at 99.5%

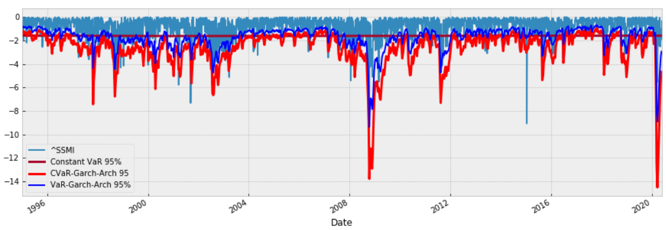

Figure 36: VaR and ES of SMI with t-student distribution at 95%

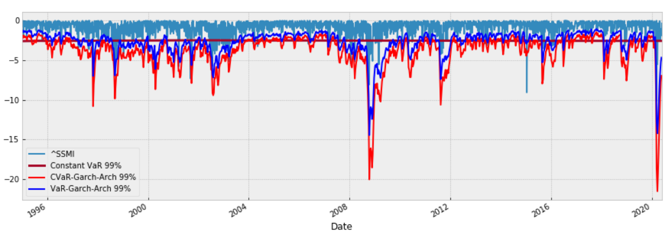

Figure 37: VaR and ES of SMI with t-student distribution at 99%

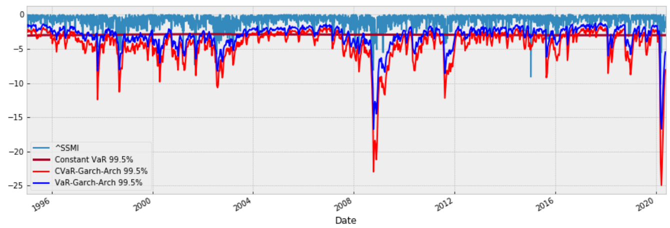

Figure 38: VaR and ES of SMI with t-student distribution at 99.5%

It is clear that the CVaR captures heaviest tails shape in the left tails for both S&P500 and SMI index, leading to the conclusion that ES reveals more risk in the distribution. In order to understand which risk of measure is more adapted, it is useful to look at the violations which each measure takes. The observed number of violations are reported in the following table for each ES and VaR for each distribution.

	Daily VaR	Expected violations
EVT Conditional	291	325
EVT Unconditional	375	325
Normal	344	319
T-student	367	319

Table 7: Violations for SMI at 95% quantile

	Daily VaR	Expected violations
EVT Conditional	46	65
EVT Unconditional	89	65
Normal	100	65
T-student	73	64

Table 8: Violations for SMI at 99% quantile

	Daily VaR	Expected violations
EVT Conditional	29	32
EVT Unconditional	50	32
Normal	68	32
T-student	40	32

Table 9: Violations for SMI at 99.5% quantile

	Daily VaR	Expected violations
EVT Conditional	245	320
EVT Unconditional	469	320
Normal	280	320
T-student	303	320

Table 10: Violations for S&P500 at 95% quantile

	Daily VaR	Expected violations
EVT Conditional	46	64
EVT Unconditional	47	64
Normal	104	64
T-student	78	64

Table 11: Violations for S&P500 at 99% quantile

	Daily VaR	Expected violations
EVT Conditional	24	32
EVT Unconditional	65	32
Normal	67	32
T-student	42	32

Table 12: Violations for S&P500 at 99.5% quantile

Testing of the SP500		
Tests or Distributions	Christoffersen Log Ratio	Binomial p-val
Student-t 95	1.428	0.35
Student-t 99	15.74	0.067
Student-t 99.5	12.115	0.076
Normal 95	7.18	0.02
Normal 99	31.16	0
Normal 99.5	45.32	0
Condit EVT 95	22.96	0
Condit EVT 99	16.92	0.04
Condit EVT 99.5	18.13	0.18
Uncondit EVT 95	86.19	0
Uncondit EVT 99	47.75	0
Uncondit EVT 99.5	43.08	0

In the above table, we can find the p-values for the Binomial test as well as the Christoffersen Log-ratio for the S&P500 index. As we can see, the null hypothesis, for the binomial test on S&P500, the null hypothesis is rejected in 8 cases with 95% confidence. We always reject the null hypothesis with the normal distribution. We also always reject with the conditional EVT except at 99.5% as the test statistic is 0.18 which is higher than 0.05. Finally, we also always reject for the unconditional EVT. The student-t distribution is the only one with which we never reject the null hypothesis.it means that this model seems well adapted and not too conservative or too flawed. As for the Christoffersen test(the convergence coverage test), we have the following critical values for a Chi-square with 2 degrees of freedom:

Critical values for a Chi-square with 2 degrees of freedom			
quantiles	0.95	0.99	0.995
Critical values	5.991	9.210	10.597

For the student-t we reject for the 99th and 99.5th percentiles as their test statistics are higher than the critical value(15.74 and 12.115 respectively),the VaR interval forecasts and the probability of exception may be respectively not independent and/or different than theta .But for the 95th quantile the Log ratio is unsignificant at the 95th percentile. With the normal distribution we always reject the null hypothesis only for the 99th and the 99.5th.For the 95th quantile,the CC test is also unsignificant at 99th percentile. For the conditional EVT and the unconditional EVT we get the same results as for the normal: We always reject the null hypothesis as the test statistics are higher than critical values.

From these tests, we can conclude that with the S&P500 index the Student-t distribution VaR estimation at 95th quantile is the best.It seems that the model is not flawed and it's estimations are independent,it's theta(1-q) observed is close to the same theta.Moreover it seems that the model fits well the datas, we are not too permissive or too conservative on the Var

Testing of the SMI index		
Tests or Distributions	Christoffersen Log Ratio	Binomial p-val
Student-t 95	7.805	0.818
Student-t 99	11.329	0.256
Student-t 99.5	6.865	0.154
Normal 95	7.429	0.117
Normal 99	19.294	0
Normal 99.5	33.917	0
Condit EVT 95	7.669	0.722
Condit EVT 99	11.231	0.3457
Condit EVT 99.5	12.731	0.033
Uncondit EVT 95	51.16	0.0024
Uncondit EVT 99	60.75	0.0024
Uncondit EVT 99.5	74.86	0.0069

We have done the same tests with the SMI index. With the binomial, we have 6 cases where the null hypothesis are rejected. We always reject when using a normal distribution, except with 95% as we get a test statistic of 0.117 which is higher than 0.05. As for the conditional EVT we only reject once at 99.5% with a test statistic of 0.033 which is lower than 0.05. Finally, we always reject with the unconditional EVT as the test statistics are always lower than 0.05.For the Conditional we always reject the VaR except at the 95th quantile. As for the Christoffersen test, we almost always reject the null hypothesis as the test statistics are higher than the critical values for the Chi-square with 2 degrees of freedom shown earlier. The only times we can not reject the null hypothesis is with the Conditional Student-t at 99.5% as its test statistic is 6.865 which is lower than 9.210(the critical value at 99%),the Conditional EVT at 95%(7.669),the Conditional Normal at 95%(7.429) and the Conditional Student-t at 99.5%(6.865) We can conclude the same thing with the SMI index. The Student-t distribution fits best with the models we test.

For the SMI we do have much more usable VaR models.the conditional student-t 95th percentile,he conditional student-t 99th percentile,the conditional normal 95th percentile,and the conditional EVT 95th percentile.Here the best VaR model to use seems to be the conditional student-t 95th percentile as it has the highest p-value under the binomial test with practically the same log ratios than the others.

Quantile Ratio (ES / VaR) for the SMI index			
Tests or Quantile Ratio	95%	99%	99.5%
GPD	1.34	1.20	1.17
Normal	1.21	1.13	1.11
Student	1.40	1.32	1.30

We can see from this table, using the S&P500index, the ratio is larger with a GPD or a Student than with a Normal distribution. We can also see that the ratio is non negligibly larger than the asymptotic value. This means that using a normal distribution would lead to an underestimation of the expected shortfall.

Quantile Ratio (ES / VaR) for the SMI index			
Tests or Quantile Ratio	95%	99%	99.5%
GPD	1.38	1.27	1.24
Normal	1.13	1.05	1.03
Student	1.48	1.32	1.30

When we observe the table with the SMI index, it becomes even more clear. The ratio for the normal distribution goes really close to 1 when we go from 95% to 99.5%. We can make the same conclusion as with the S&P500. The ratio under the normal distribution converges to 1 whereas the other two stay far. This shows that scaling quantiles with the asymptotic ratio leads to an underestimation of the expected shortfall.

Bootstrap test whether the overage residuals are equal to 0 (6mln observations)			
pvalues/Quantile Ratio	95%	99%	99.5%
SP500	0.25	0.99	1
SMI	0.12	0.814	0.69

We observe under the bootstrap test that all the overage residuals in the GPD are not significantly different than 0 under all the index and quantiles across them.

3 Conclusion

In this paper, we have used the GARCH model in combination with extreme-value theory to solve the problem of non-normality of S&P500 and SMI index log returns. We have also compared the use of ES and VaR through different type of distributions. This allows us to conclude that the student distribution fits better in order to estimate innovations as it is able to take into account larger tails, opposed to the normal distribution which underestimates the extreme values. However, the student distribution is limited as this only works if the tails are symmetric. This is where the GDP comes in superior as it can handle the asymmetry in the tails. The results gained from the A.J. McNeil and R. Frey paper are similar to this study. Starting from the residuals, their distributions are most of the time with fat tails, which follows that they are consistently far away from the normality.

The biggest point of this project was to understand the importance of the risk management in finance. Especially by understanding how the way risk behaves in the extreme cases. To see that, we had to examine the left tail of the distribution of the return. The ways to do it are known as Value at Risk (VaR) and Expected Shortfall (ES). And our results shows that McNeil, A. J., and R. Frey were precursors for the time in their field of work.

4 References

McNeil, A. J., and R. Frey. 2000. Estimation of tail-related risk measures for heteroskedastic financial time series: An extreme value approach. Journal of Empirical Finance7:271–300.